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COMMENT

Exact analytical eigenfunctions for the $x^2+\lambda x^2/(1+gx^2)$ interaction

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Abstract. We discuss an ansatz for the eigenfunctions of the $x^2+\lambda x^2/(1+gx^2)$ interaction recently introduced by Blecher and Leach. We show that the ansatz provides pairs of solutions in five cases having λ and g connected by $\lambda=-(6g^2+4g)$ and conjecture the possible existence of an infinite number of such solution pairs.

1. Introduction

In a recent paper in this journal Blecher and Leach (1987) discussed the eigenproblem

$$\psi''+[\varepsilon-x^2-\lambda x^2/(1+gx^2)]\psi=0 \tag{1.1}$$

for positive g and for x in the interval $(-\infty,\infty)$. This potential is of interest in laser physics (as the reduction of the Fokker-Planck equation of a single-mode laser under suitable conditions) and in elementary particle physics (as a one-dimensional Schrödinger equation associated with a zero-dimensional field theory). Also, the three-dimensional analogue of (1.1) was recently found by Varshni (1987) to produce a sequence of energy levels which is identical to that occurring in the shell model of the nucleus. Apart from these applications, the eigenproblem (1.1) has been extensively used as a test ground for several approximate methods to generate the energy eigenspectrum. (For specific references see the paper of Blecher and Leach and the references therein.) In this respect the paper by Blecher and Leach (hereafter referred to as BL) is interesting since it reports the existence of *exact analytical* eigensolutions to the problem (1.1) in the form

$$\psi(x)=\exp(-\tfrac{1}{2}x^2)\,(1+gx^2)\sum_{n=0}^{N}c_nx^{2n+\delta} \tag{1.2}$$

for suitable choices of λ and g. They reported three cases for which it should be possible to simultaneously obtain two exact analytical eigensolutions of (1.1).

The present comment is motivated by our finding that, surprisingly, none of the functions reported by BL actually solves the simultaneous eigenproblem. A reinvestigation of the subject shows that, after corrections, one pair of functions reported by BL is indeed a solution but the other two pairs are definitely not. Furthermore, our reinvestigation shows that it is always possible to find an equation for g defining the existence, or not, of pairs of solutions. As a by-product we report new pairs of solutions. Our results seem to indicate the possible existence of an infinite number of solution pairs having λ and g connected by the relation $\lambda=-(6g^2+4g)$.

2. The results of Blecher and Leach

In this section we want to reinvestigate the cases considered by Blecher and Leach in § 3 of their paper. We start by discussing their equations (3.5) and (3.7). It is easy to verify that neither ψ_0 nor ψ_3 solve their (1.1) with the potential (3.5). However, (3.5) contains $\lambda=-\frac{1}{18}$ although for $g=\frac{1}{6}$, according to their (3.4), one should use $\lambda=-\frac{13}{18}$. Using $\lambda=-\frac{13}{18}$ in (3.5) it is easy to verify that ψ_0 is a solution of (1.1) while ψ_3 is not. This means that the coefficient $-\frac{7}{9}$ in (3.7) must be incorrect. The next question is whether a different coefficient would produce the desired solution. To answer this question we substitute both

$$\psi_a=(1+gx^2)\exp(-\tfrac{1}{2}x^2) \tag{2.1a}$$

$$\psi_b=x(1+ax^2)(1+gx^2)\exp(-\tfrac{1}{2}x^2) \tag{2.1b}$$

into (1.1), multiply by $1+gx^2$ and from the coefficients of the several powers of x obtain the following *non-linear* system of equations:

$$g\varepsilon_1-5g-\lambda=0 \tag{2.2a}$$

$$2g+\varepsilon_1-1=0 \tag{2.2b}$$

$$g\varepsilon_2-11g-\lambda=0 \tag{2.2c}$$

$$a(20g+\varepsilon_2-7)+g\varepsilon_2-7g-\lambda=0 \tag{2.2d}$$

$$6a+6g+\varepsilon_2-3=0. \tag{2.2e}$$

From (2.2*a*, *c*), (2.2*a*, *b*) and (2.2*b*, *e*) one obtains, respectively,

$$\varepsilon_2=\varepsilon_1+6 \tag{2.3a}$$

$$\lambda=-(2g^2+4g) \tag{2.3b}$$

$$\varepsilon_1=3a+3. \tag{2.3c}$$

Substituting these results back into (2.2) yields the following two equations:

$$2g+3a+2=0 \tag{2.3d}$$

$$23ag+3a^2+2g^2+2a+6g=0. \tag{2.3e}$$

Obviously, (2.2) and (2.3) are perfectly equivalent. Now it is trivial to check that the values $g=\frac{1}{6}$, $\lambda=-\frac{13}{18}$, $\varepsilon_1=\frac{2}{3}$, $\varepsilon_2=\frac{20}{3}$ and $a=-\frac{7}{9}$ obtained by Blecher and Leach fulfil (2.3*a*–*d*) but fail to satisfy (2.3*e*). Substituting a from (2.3*d*) into (2.3*e*) leads to the equation

$$g(3g+2)=0. \tag{2.4}$$

This equation shows that there is no solution having $g>0$ such that (2.1*a*) and (2.1*b*) are simultaneously eigenfunctions of (1.1). Blecher and Leach did not obtain the results above because they used $-3g$ instead of the correct value, $-13g$, in the expression for ε_2 in their (3.3).

We now proceed to investigate (3.10) and (3.12). First, note that in both equations x^3 should be x^2. Writing

$$\psi_a=x(1+gx^2)\exp(-\tfrac{1}{2}x^2) \tag{2.5a}$$

$$\psi_b=(1+ax^2+bx^4)(1+gx^2)\exp(-\tfrac{1}{2}x^2) \tag{2.5b}$$

we obtain the following system of non-linear equations:

$$g\varepsilon_1 - 7g - \lambda = 0 \tag{2.6a}$$

$$6g + \varepsilon_1 - 3 = 0 \tag{2.6b}$$

$$b(-g\varepsilon_2 + 13g + \lambda) = 0 \tag{2.6c}$$

$$a(\lambda - g\varepsilon_2 + 9g) - b(30g + \varepsilon_2 - 9) = 0 \tag{2.6d}$$

$$a(12g + \varepsilon_2 - 5) + 12b + g\varepsilon_2 - 5g - \lambda = 0 \tag{2.6e}$$

$$2a + 2g + \varepsilon_2 - 1 = 0. \tag{2.6f}$$

For $b = 0$ (as in (3.10) of BL) the above system reduces to

$$\varepsilon_2 = \varepsilon_1 + 2 \tag{2.7a}$$

$$\lambda = -(6g^2 + 4g) \tag{2.7b}$$

$$\varepsilon_1 = -3a - 3 \tag{2.7c}$$

$$a + 2 - 2g = 0 \tag{2.7d}$$

$$9ag - 3a^2 - 6a - 2g + 6g^2 = 0. \tag{2.7e}$$

The last two equations imply

$$g(3g - 2) = 0. \tag{2.8}$$

Therefore $g = \frac{2}{3}$ is the only non-trivial solution. It requires $\lambda = -\frac{16}{3}$ rather than $\lambda = -\frac{8}{3}$ used by BL in their (3.8).

For $b \neq 0$, (2.6) reduces to

$$\varepsilon_2 = \varepsilon_1 + 6 \tag{2.9a}$$

$$\lambda = -(6g^2 + 4g) \tag{2.9b}$$

$$\varepsilon_1 = -3a - 9 \tag{2.9c}$$

$$a = 2g - 4 \tag{2.9d}$$

$$b = (2 - g)/3 \tag{2.9e}$$

$$3g^2 - 3g - 2 = 0 \tag{2.9f}$$

implying as the only positive solution $g = \frac{1}{2} + \sqrt{\frac{11}{12}} \simeq 1.457\,43$. In this case we have $a \simeq -1.0851$ and $b \simeq 0.1809$ and the eigensolution has four zeros. Therefore ψ_a and ψ_b correspond to the first and fourth excited states, respectively, and not first and second as found by BL.

3. New results

Using the procedure described in § 2 we searched for further pairs of simultaneous eigensolutions. Writing

$$\psi_a = x(1 + gx^2)\exp(-\tfrac{1}{2}x^2) \tag{3.1a}$$

$$\psi_b = (1 + ax^2 + bx^4 + cx^6 + dx^8 + ex^{10})(1 + gx^2)\exp(-\tfrac{1}{2}x^2) \tag{3.1b}$$

we found the following new solutions:

$$\varepsilon_1 = 3 - 6g \tag{3.2a}$$

$$\varepsilon_2 = 1 - 2g - 2a \tag{3.2b}$$

$$\lambda = -(6g^2 + 4g). \tag{3.2c}$$

(i) For $d = e = 0$:

$$a = 2g - 6 \tag{3.3a}$$

$$b = (12 + 2g - 3g^2)/3 \tag{3.3b}$$

$$c = -2b/25 \tag{3.3c}$$

$$45g^3 - 57g^2 - 92g - 12 = 0. \tag{3.3d}$$

The roots of (3.3d) are 2.234 85, −0.823 25 and −0.144 94. In this case ψ_b corresponds to the sixth excited state since it cuts the x axis at $x^2 \simeq 8.133\,26$, 3.473 43 and 0.893 308.

(ii) For $e = 0$:

$$a = 2g - 8 \tag{3.4a}$$

$$b = (24 + 2g - 3g^2)/3 \tag{3.4b}$$

$$c = (36g^3 - 48g^2 - 152g - 96)/45 \tag{3.4c}$$

$$d = -c/21 \tag{3.4d}$$

$$675g^4 - 1017g^3 - 2784g^2 - 796g - 48 = 0. \tag{3.4e}$$

The roots of (3.4e) are 3.009 79, −0.0844, −0.2368 and −1.1818. ψ_b corresponds to the eighth excited state, cutting the axis at $x^2 \simeq 11.4695$, 6.055 06, 2.741 96 and 0.733 472.

(iii) Considering all terms in (3.1b):

$$a = 2g - 10 \tag{3.5a}$$

$$b = (40 + 2g - 3g^2)/3 \tag{3.5b}$$

$$c = (12g^3 - 18g^2 - 84g - 80)/15 \tag{3.5c}$$

$$d = (80 + 284g + 528g^2 + 123g^3 - 75g^4)/105 \tag{3.5d}$$

$$e = -2d/63 \tag{3.5e}$$

$$-11\,025g^5 + 19\,107g^4 + 76\,098g^3 + 34\,572g^2 + 4488g + 160 = 0. \tag{3.5f}$$

The roots are 3.783 83, −1.5381, −0.3228, −0.1305 and −0.059 13. ψ_b is the tenth excited state, cutting the axis at $x^2 \simeq 14.8768$, 8.800 16, 4.930 12, 2.271 17 and 0.621 754.

4. Conclusion

The eigenproblem (1.1) admits solution pairs that can be expressed analytically as the product of $(1 + gx^2)\exp(-\frac{1}{2}x^2)$ times a single polynomial as indicated in (1.2). These pairs consist of one odd eigenfunction as given by (3.1a) and one even eigenfunction as given by (3.1b). We presented explicit solutions for the first five cases. These cases always had λ and g connected by $\lambda = -(6g^2 + 4g)$. The possible values of g are 0.666 666, 1.457 43, 2.234 85, 3.009 79 and 3.783 83. These values are curiously spaced by an almost constant difference Δg: 0.7908, 0.7775, 0.7749 and 0.7740. The maximum power of x in the last factor of (1.2) gives the quantum number of the state. Our results suggest the possible existence of an infinite number of solution pairs obeying $\lambda = -(6g^2+4g)$. It should be interesting to prove (or disprove) this and to check whether Δg converges to a constant.

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